Application of Extended Element-Free Galerkin Method to Nonlinear Problem

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Abstract — A new method has been proposed for implementing essential boundary conditions to the Element-Free Galerkin Method (EFGM) without using the Lagrange multiplier. The basic idea of the method is to use interpolation functions of a small support radius. The performance of the proposed method has been investigated for the nonlinear Poisson problem. The results of computations show that, as interpolation functions become closer to delta functions, the accuracy of the solution is improved on the boundary. In addition, by means of the proposed method, the accuracy of the solution is also improved in the vicinity of the boundary. Therefore, it might be concluded that the proposed method is useful for solving the nonlinear Poisson problem.

I. INTRODUCTION

Many meshless approaches have been proposed and have yielded excellent results in the fields of the shielding current analysis of the high-temperature superconductor [1, 2] and the analysis of the electromagnetic-wave scattering problem [3] etc. However, meshless approaches are plagued by the following difficulty: the method for implementing the essential boundary condition is different according to meshless approaches. For example, as the implementation method, the Lagrange multiplier and the penalty method are used in the Element-Free Galerkin Method (EFGM) [4] and the meshless local Petrov-Galerkin method, respectively. If a new implementation method of the essential boundary condition were proposed without dependence on meshless approaches, the above demerit could be completely resolved.

The purpose of the present study is to propose a new method for implementing the essential boundary condition to the EFGM and to evaluate the performance of the proposed method by comparing with the conventional method.

II. EXTENDED ELEMENT-FREE GALERKIN METHOD

In the present study, we consider a 2D nonlinear Poisson problem with the inhomogeneous term $\rho(u)$ which has a nonlinear dependence on the dependent variable. By using the successive substitution method, the solution u is determined by means of the iterative procedure. In the kth step, we solve the following linear problem for $u^{(k+1)}$:

$$-\Delta u^{(k+1)} = \rho(u^{(k)}) \quad \text{in } \Omega, \tag{1}$$

$$u^{(k+1)} = \overline{u} \quad \text{on } \Gamma_{\mathrm{D}}, \tag{2}$$

$$q^{(k+1)} \equiv \partial u^{(k+1)} / \partial n = \overline{q} \quad \text{on } \Gamma_{\mathrm{N}}, \tag{3}$$

where Ω denotes a region bounded by a simple closed curve $\partial\Omega$ and the curve satisfies the following relations: $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}} = \partial\Omega$ and $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \phi$. Furthermore, $\rho(u)$, $\overline{u}$ and $\overline{q}$ are known functions in Ω, on Γ_{D} and on Γ_{N}, respectively. In addition, $\boldsymbol{n}$ and the superscript (k) indicate an outward unit normal on $\partial\Omega$ and an iteration number label, respectively. The above step is repeated until $\left\|u^{(k+1)} - u^{(k)}\right\| / \left\|u^{(k+1)}\right\| \le 10^{-10}$ is satisfied.

In the classical EFGM, the essential boundary condition is incorporated into a weak form through Lagrange multiplier. In this section, we explain a new method for implementing the essential boundary condition to the EFGM without using the Lagrange multiplier.

Let us first discretize the essential boundary condition (2). Apparently, (2) is equivalent to the proposition:

$$\int_{\Gamma_{\mathrm{D}}} p(s)\left[u^{(k+1)} - \overline{u}\right] d\ell = 0, \tag{4}$$

for $\forall p$. When the function p is assumed to be contained in the functional space $U \equiv \mathrm{span}(N_1, N_2, \cdots, N_M)$, we get

$$p(s) = \sum_{i=1}^{M} N_i(s)\, \hat{p}_i, \tag{5}$$

where s denotes an arclength along $\partial\Omega$. Furthermore, $\hat{p}_i$ $(i = 1, 2, \cdots, M)$ are all constants. By using (5), (4) can be discretized as

$$C^T \hat{\boldsymbol{u}}^{(k+1)} = \boldsymbol{g}. \tag{6}$$

Here, C and $\boldsymbol{g}$ are defined by

$$C = \sum_{i=1}^{N}\sum_{j=1}^{M}\left(\int_{\Gamma_{\mathrm{D}}} \phi_i\, N_j\, d\ell\right)\boldsymbol{e}_i \boldsymbol{e}_j{}^T, \tag{7}$$

$$\boldsymbol{g} = \sum_{i=1}^{M}\left(\int_{\Gamma_{\mathrm{D}}} N_i\, \overline{u}\, d\ell\right)\boldsymbol{e}_i, \tag{8}$$

where ϕ_i's are shape functions obtained by the moving least-square approximation.

Next, the weak form can be similarly discretized as

$$\forall \hat{\boldsymbol{w}} \text{ s.t. } C\hat{\boldsymbol{w}} = 0 : \hat{\boldsymbol{w}}^T\left[A\hat{\boldsymbol{u}}^{(k+1)} - \boldsymbol{f}^{(k)}\right] = \boldsymbol{0}, \tag{9}$$

$$\Leftrightarrow \exists \hat{\boldsymbol{\lambda}}^{(k+1)} \in \boldsymbol{R}^M : A\hat{\boldsymbol{u}}^{(k+1)} + C\hat{\boldsymbol{\lambda}}^{(k+1)} = \boldsymbol{f}^{(k)}.$$

Here, A and $\boldsymbol{f}^{(k)}$ are defined by

$$A = \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\iint_{\Omega} \nabla\phi_i \cdot \nabla\phi_j\, d\boldsymbol{x}^2\right)\boldsymbol{e}_i \boldsymbol{e}_j{}^T, \tag{10}$$

$$\boldsymbol{f}^{(k)} = \sum_{i=1}^{N}\left(\iint_{\Omega} \phi_i\, \rho(u^{(k)})\, d\boldsymbol{x}^2 + \int_{\Gamma_{\mathrm{N}}} \phi_i\, \overline{q}\, d\ell\right)\boldsymbol{e}_i. \tag{11}$$

By combining (6) with (9), we can obtain a linear system. In this way, the nonlinear Poisson problem is reduced to the problem in which the linear system is solved. Throughout the present study, the above method is called the eXtended Element-Free Galerkin Method (X-EFGM).

As mentioned above, p is an arbitrary function. Therefore, $N_i(s)$ can be selected arbitrarily. For example, if $N_i(s)$ is given by the Lagrange interpolant, the X-EFGM becomes equivalent to the classical EFGM. In the present study, a function $N_i(\xi)$ is given by $N_{\sigma(e,j)}(\xi) = \left(2/\bar{\delta}\, l_e\right)\Psi_j(\xi)$ (see Fig.1). Here, $\sigma(e, j)$ and l_e are the global number of the jth local node in the eth integration cell and the length of eth integration cell, respectively.

III. NUMERICAL RESULTS

In this section, we investigate the performance of the X-EFGM by comparing with the classical EFGM. Throughout the present study, the target region Ω is given by $\Omega \equiv (0, 1) \times (0, 1)$ and $\rho(u)$ is assumed as $\rho(u) = \left(u^{-3} + u\right)/2$. Obviously, the analytic solution of the nonlinear Poisson problem is given by $u = \sqrt{\sin(x+y)}$. In addition, we adopt a particular solution of $-\Delta u = 0$ as the initial solution $u^{(1)}$ and its explicit form is written as $u^{(1)} = -\cosh x \sin y + \cos x \sinh y$.

Let us first investigate the influence of $N_i(\xi)$ on the accuracy of the X-EFGM for the Dirichlet problem. As the measure of the accuracy, we adopt the relative error defined by

$$\varepsilon_{\mathrm{B}} = \frac{\sqrt{\int_{\partial\Omega}\left[u(\boldsymbol{x}(s)) - \bar{u}\right]^2 d\ell}}{\sqrt{\int_{\partial\Omega}\bar{u}^2\, d\ell}}. \tag{12}$$

The relative error of the X-EFGM is calculated as a function of the dimensionless support radius $\bar{\delta}$ and is depicted in Fig. 2. We see from this figure that the relative error decreases monotonously with a decrease in $\bar{\delta}$. This tendency does not depend on the total number N of nodes. The above result indicates that the accuracy of the X-EFGM becomes high for the case with $N_i(\xi) \approx \delta(\xi - \xi_i)$. For this reason, $N_i(\xi)$ is fixed as $N_i(\xi) = \delta(\xi - \xi_i)$, hereafter.

Next, we investigate both the accuracy of the X-EFGM and that of the classical EFGM. As the measure of the accuracy of the numerical solution, we adopt the error:

$$\varepsilon_{\mathrm{D}}(\boldsymbol{x}) \equiv \frac{\left|u_{\mathrm{A}}(\boldsymbol{x}) - u_{\mathrm{N}}(\boldsymbol{x})\right|}{\underset{\boldsymbol{x}}{\mathrm{Max}}\left|u_{\mathrm{A}}(\boldsymbol{x})\right|}, \tag{13}$$

where subscript notations, A and N, are analytic and numerical solutions, respectively. Typical examples of the error distribution are shown in Figs. 3(a) and 3(b). We see from these figures that the accuracy of the X-EFGM is higher than that of the classical EFGM in the vicinity of $\partial\Omega$.

From these results, it might be concluded that the X-EFGM is useful for solving the nonlinear Poisson problem.

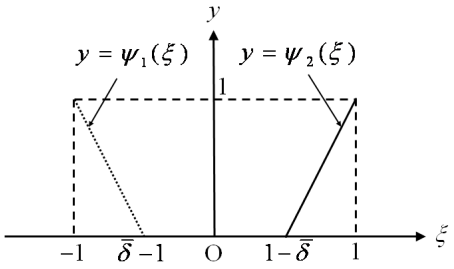

Fig. 1. The graphs of $\Psi_1(\xi)$ and $\Psi_2(\xi)$.

IV. CONCLUSION

By using a new method for implementing the essential boundary condition to the EFGM, we have developed the X-EFGM. In addition, we have investigated its performance by comparing with the classical EFGM. Conclusions obtained in the present study are summarized as follows.

1) When the value of $\bar{\delta}$ decreases, the accuracy of the numerical solution is drastically improved on $\partial\Omega$.
2) In the X-EFGM, the accuracy of the numerical solution is also improved in the vicinity of $\partial\Omega$.

V. REFERENCES

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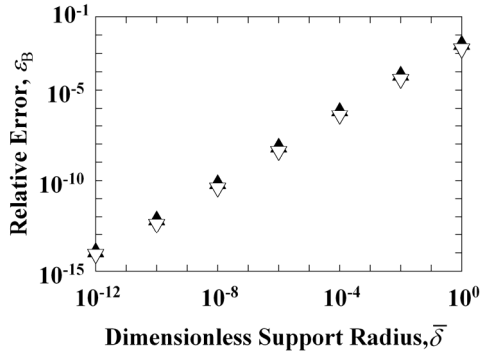
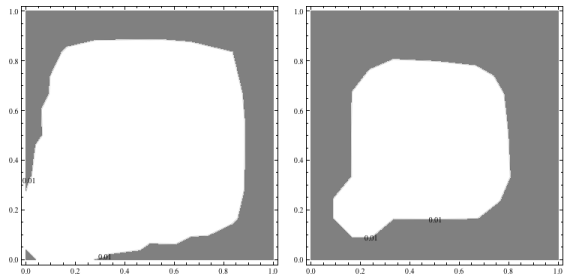


Fig. 2. Dependence of the relative error ε_{B} on the dimensionless support radius $\bar{\delta}$. Here, ▲ : N = 81 and ▽ : N = 529.



(a) Classical EFGM (b) X-EFGM

Fig. 3. The error distribution for the case with (a) the classical EFGM and (b) the X-EFGM (N = 49). The gray region indicates that $\varepsilon_{\mathrm{D}}(\boldsymbol{x}) \le 10^{-2}$ is satisfied.